Synchronization of Phase-coupled Oscillators with Distance-dependent Delay

Karol Trojanowski* and Lech Longa†

Marian Smoluchowski Institute of Physics,

Department of Statistical Physics and Mark Kace Center for Complex Systems Research,

Jagiellonian University, Reymonta 4, Kraków, Poland

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Abstract

By means of numerical integration we investigate the coherent and incoherent phases in a generalized Kuramoto model of phase-coupled oscillators with distance-dependent delay. Preserving the topology of a complete graph, we arrange the nodes on a square lattice while introducing finite interaction velocity, which gives rise to non-uniform delay. It is found that such delay facilitates incoherence and removes reentrant behavior found in models with uniform delay. A coupling-delay phase diagram is obtained and compared with previous results for uniform delay.

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* e-mail address:karol.trojanowski@uj.edu.pl

† e-mail address:lech.longa@uj.edu.pl

I. MODEL SUMMARY

A. The Kuramoto model

The popular Kuramoto model of mutual synchronization of coupled oscillators [1] has, since its inception, drastically improved the understanding of this prevalent phenomenon. Common examples [2][3] include synchronous chirping of crickets, flashing of Chinese fireflies [4], clapping of audiences, bursting of neurons, contraction of heart muscles or operation of Josephson junction arrays [5][6] to name a few. This model still remains the most succesful one, due to its mathematical tractability, combined with the ability to capture the essence of synchrony.

We build up from the definition of the Kuramoto model which is most suitable for direct treatment by numerical methods [2][7]:

$$\dot{\theta}_i(t) = \omega_i + \frac{K}{N} \sum_{j=1}^{N} \sin(\theta_j(t) - \theta_i(t)) \tag{1}$$

where $i = 1 \ldots N$, $\theta_i(t)$ is the phase of the i-th oscillator at time t and ω_i are intrinsic oscillator frequencies, sampled from yet unspecified probability distribution $\rho(\omega)$ on compact support. Kuramoto solved this model exactly in the case of $N \to \infty$ and ω_i sampled from a Lorentz distribution. Solutions for other distributions have subsequently been obtained. A model such defined exhibits a (mean-field-type) phase transition between the disordered (incoherent) and ordered (coherent) phases as the coupling constant K is increased. Order is monitored by the real parameter r, defined as:

$$r(t)e^{i\psi(t)} = \frac{1}{N} \sum_{j=1}^{N} e^{i\theta_i(t)}. \tag{2}$$

When the stationary state is assumed, $r(t) = r$ ($r\epsilon[0,1]$), with $r = 1$ and $r = 0$ in total coherence and incoherence, respectively.

B. Introducing delay

Some real systems cannot be considered without taking delay into account. The popular example of a clapping audience synchronizing to clap in unison is valid only for sufficiently small audiences, such as opera halls. When distances are of the order of $300m$, or higher,

the finite speed of sound makes the delay non-negligible. As a result, e.g. football arena audiences cannot clap together or have difficulty in coherent singing.

We start by introducing delay to (1) in the most general way:

$$\dot{\theta}_i(t) = \omega_i + \frac{K}{N} \sum_{j=1}^{N} \sin(\theta_j(t - \tau_{ij}) - \theta_i(t)). \tag{3}$$

The case of uniform delay, $\tau_{ij} \equiv \tau$, is interpreted as coupling of the state at t to the state at $t - \tau$. The stability of incoherence in such a model has been studied by Yeung and Strogatz in [8].

To introduce non-uniform delay, we arrange the nodes on a square lattice while preserving the topology of a complete graph. The coupling remains uniform, however the delay is made distance-dependent through the definition:

$$\tau_{ij} = \tau \cdot \frac{s_{ij}}{\langle s_{ij} \rangle}. \tag{4}$$

τ is interpreted as the inverse velocity. The distance s_{ij} is defined with the so-called "taxi-driver's measure", i.e. as sum of the differences in horizontal and vertical coordinates and is measured in number of nodes. To maintain translational invariance we identify the opposite edges and the shortest route is always preferred. Hence, when $N = L \times L$, the average distance between any pair of nodes $\langle s_{ij} \rangle = \frac{L}{2}$. This definition normalizes the maximum delay to 2τ and removes dependence on network size. We find that the unmodified parameter r, defined by (2), is useful in monitoring the average order in the sample.

II.   SIMULATIONS AND RESULTS

We have investigated the behavior of the model described by (3) and (4) by integrating the equations (3) using a four-step Adams-Bashforth scheme. For simplicity and reference with previous results [8], we set $\omega_i = \frac{\pi}{2}$ for all i, therefore $\rho(\omega) = \delta(\omega - \frac{\pi}{2})$. A run for one pair of parameters (τ, K) consisted of 10000 integration steps with step size $\Delta t = 0.01$, out of which the last 6000 were considered for averaging the order parameter r to obtain the temporal average $\overline{r}$, rejecting the first 4000 when the system is approaching stability. Lattices as large as 32×32 were considered. The initial conditions, as well as histories of θ_i, were sampled uniformly from $[0, 2\pi)$.
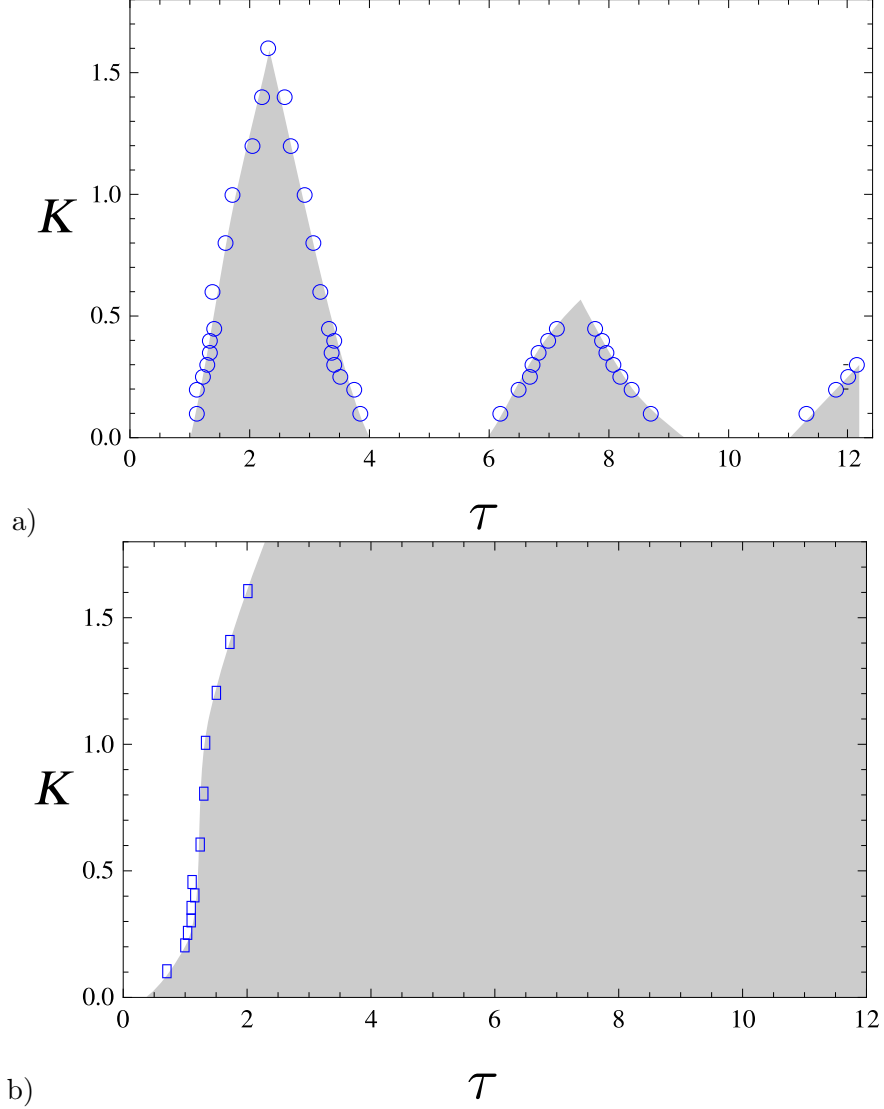
FIG. 1. Phase diagram portions for the uniform delay model (a) and the model described by (4) (b). The shaded areas visually approximate the incoherent regimes. The phase border points are results of numerical integration of (3).

We have found phase boundaries between the completely ordered and disordered phase (Fig. 1b). For reference, we have produced a diagram for $\tau_{ij} \equiv \tau$ (Fig. 1a). In the case of $\rho(\omega) = \delta(\omega - \omega_0)$, the transitions occuring with changing τ are instanteous. It is observed that non-uniform delay removes the reentrance of synchrony, as intuitively expected. This difference is easily understood when considering low coupling. The reentrance in the case of uniform delay and $K \ll 1$ is due to there being sufficiently little difference between states at $t - \tau$ and t for the effect of delay to be approximated by the rotation of all oscillators

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with average frequency:

$$\sin(\theta_j(t-\tau) - \theta_i(t)) \approx \sin(\theta_j(t) - \theta_i(t) - \overline{\omega}\tau) \tag{5}$$

The reentrance for low coupling then occurs when $\overline{\omega}\tau$ is close to an integer multiple of 2π ($\tau \approx k\frac{2\pi}{\overline{\omega}}$, $k = 1, 2, \ldots$), where the low-delay limit is reproduced. Our simulations reflect this heuristic quantitatively up to $k = 2$ and qualitatively from $k = 3$ on. In the case of distance-dependent delay (Fig. 1b) and $K \ll 1$ the effects of delay are individually approximated by phase shifts of $\overline{\omega}\tau\frac{s_{ij}}{\langle s_{ij}\rangle}$ which vary across connections and the low-delay limit cannot be reproduced by a specific value of delay. Hence, no reentrance occurs.

III.   CONCLUSIONS

Our results prove that in order to realistically reproduce behavior of synchronizable systems in which delay cannot be neglected, the dependence of delay on distance must also be accounted for. However, only this dependence was considered in this research, namely the case that even distant nodes get to affect other distant nodes with the same strength as they affect their closest neighbors, in consistence with the mean-field approximation. This is indeed true for some systems, such as digital communication networks. The effect on the phase diagram when coupling decreasing with distance is taken into account needs to be considered in due course.

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